THE SECOND BETTI NUMBER OF DOUBLY WEIGHTED HOMOLOGY GROUPS OF A CERTAIN PRE LIE SUPERALGEBRA

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[1] [2] [3]

Abstract

The space of multi-vector fields on a manifold has a structure of $\mathbb{Z}$-graded Lie superalgebras under the Schouten bracket. (We call such an algebra a pre Lie superalgebra.) As in the case of Lie algebras, we have the associated homology groups similarly. We treat the case where the underlying manifold is a Euclidean n-space, and the coefficients are polynomials. Under such conditions, we show the vanishing of the second homology group of the chain complexes which are doubly indexed by the pairs of integers (w, h), which we call weights. We specially treat the case $w = h$ in this article.

Introduction

There is a notion of (weighted) (co)homology group theory of pre Lie superalgebras like those of Lie algebras. In [2], we introduced the notion of doubly weighted chain spaces and doubly weighted homology groups for doubly graded pre Lie superalgebras. The pre Lie superalgebra we handle in this paper is the exterior algebra of polynomial coefficient multi-vector fields on n-plane with the Schouten bracket as the super bracket. Each 1-chain is a linear combination of the following type elements.

$$x_1^{b_1} \cdots x_n^{b_n} (\frac{\partial}{\partial x_1})^{a_1} \wedge \cdots \wedge (\frac{\partial}{\partial x_n})^{a_n} \quad \text{(often abbreviated as } x^B \partial_A)$$

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where $B = (b_1, \ldots, b_n)$, $(b_i \geqq 0)$ and $A = (a_1, \ldots, a_n)$, $(a_i \in \{0, 1\})$. For the 1-chain $x^B \partial_A$, we call the numbers $-1 + |A|$, $-1 + |B|$, and $(-1 + |A|, -1 + |B|)$ primary weight, secondary weight, and double weight, respectively, where $|A|$ = length of $A = \sum_{i=1}^{n} a_i$ and $|B| = \sum_{i=1}^{n} b_i$ (cf. Definition 1.9 and (2.2)).

In the same situation as above, we have proven in [2] the following (1)–(4):
(1) The Euler number is 0 for all doubly weighted homology groups.
(2) Each Betti number is 0 for (w, h)-doubly weighted homology groups if $w \neq h$.
(3) The first Betti number is 0 for all doubly weighted homology groups.
(4) The second Betti number is 0 for doubly weighted homology groups if $w = h = 0$.

In this paper, we prove
(5) The second Betti number is 0 for every doubly weighted homology groups with $w = h$.

Combining (2) and (5), we obtain
(6) The second Betti number is 0 for all doubly weighted homology groups.

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1 Preliminaries

We summarize the notions and notations that are necessary for the discussion. First, we recall the definitions.

Definition 1.1 (pre Lie superalgebra). Suppose $\mathfrak{g}$ is a vector space over $\mathbb{R}$ not necessarily finite-dimensional, graded over $\mathbb{Z}$ as $\mathfrak{g} = \sum_{j \in \mathbb{Z}} \mathfrak{g}_j$ and have a bilinear operation $[.,.]$ satisfying

(1.1) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$

(1.2) $[X, Y] + (-1)^{xy}[Y, X] = 0$ where $X \in \mathfrak{g}_x$ and $Y \in \mathfrak{g}_y$ (super commutativity)

(1.3) $(-1)^{xz}[[X, Y], Z] + (-1)^{yx}[[Y, Z], X] + (-1)^{zy}[[Z, X], Y] = 0$ (super Jacobi id).

Then we call $\mathfrak{g}$ a pre (or $\mathbb{Z}$-graded) Lie superalgebra.

In the usual Lie algebra homology theory, m-th chain space is the exterior algebra $\Lambda^m \mathfrak{g}$ of $\mathfrak{g}$ and the boundary operator is essentially induced from $X \wedge Y \mapsto [X, Y]$.
In pre Lie superalgebras, by "super" skew-symmetry of bracket operation, m-th chain space C_m is defined so that C_m is the quotient of the tensor space $\otimes^m \mathfrak{g}$ of $\mathfrak{g}$ by the 2-sided ideal generated by

(1.4) $$X \otimes Y + (-1)^{xy} Y \otimes X \quad \text{where} \quad X \in \mathfrak{g}_x, Y \in \mathfrak{g}_y\ .$$

We denote the equivalence class of $X \otimes Y$ by $X \triangle Y$.
Note that since $X_{\mathrm{odd}} \triangle Y_{\mathrm{odd}} = Y_{\mathrm{odd}} \triangle X_{\mathrm{odd}}$ and $X_{\mathrm{even}} \triangle Y_{\mathrm{any}} = -Y_{\mathrm{any}} \triangle X_{\mathrm{even}}$ hold, $\triangle^m \mathfrak{g}_i = \overbrace{\mathfrak{g}_i \triangle \cdots \triangle \mathfrak{g}_i}^{m}$ has symmetric property for odd i and has skew-symmetric property for even i with respect to action of permutations.

Remark 1.2. We often automatically regard $\Lambda^i V$ and $a \wedge b$ have skew-symmetric property. To avoid this confusion, we use an odd notation $\triangle^i V$ or $a \triangle b$ here. Note also that $\triangle$ is only $\mathbb{R}$-bilinear.

We introduce a recursive definition of the boundary operator using the left action.

$$\partial(A_0 \triangle A_1 \triangle \cdots \triangle A_m) = -A_0 \triangle \partial(A_1 \triangle \cdots \triangle A_m) + A_0 \cdot (A_1 \triangle \cdots \triangle A_m) \tag{1.5}$$

where

$$A_0 \cdot (A_1 \triangle \cdots \triangle A_m) = \sum_{i=1}^{m} (-1)^{a_0 \sum_{s<i} a_s} A_1 \triangle \cdots \triangle [A_0, A_i] \triangle \cdots \triangle A_m \tag{1.6}$$

for each homogeneous elements $A_i \in \mathfrak{g}_{a_i}$. In lower degree cases, the boundary operator is given as below:

$$\partial(A \triangle B) = [A, B] \tag{1.7}$$

$$\partial(A \triangle B \triangle C) = -A \triangle [B, C] + [A, B] \triangle C + (-1)^{ab} B \triangle [A, C] \tag{1.8}$$

for homogeneous elements $A \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$, $C \in \mathfrak{g}_c$.
A prototype of pre Lie superalgebra is the exterior algebra of the sections of exterior power of tangent bundle of a differentiable manifold M of dimension n,

$$\mathfrak{g} = \sum_{i=1}^{n} \Gamma \Lambda^i \mathrm{T}(M) = \sum_{i=0}^{n-1} \mathfrak{g}_i \ , \quad \text{where} \quad \mathfrak{g}_i = \Gamma \Lambda^{i+1} \mathrm{T}(M) \tag{1.9}$$

with the Schouten bracket. By abuse of notation we often abbreviate $\Gamma \Lambda^i \mathrm{T}(M)$ as $\Lambda^i \mathrm{T}(M)$.
There are several ways defining the Schouten bracket, namely, axiomatic explanation, sophisticated one using Clifford algebra or more direct ones (cf. [1]). In the context of Lie algebra homology theory, we introduce the Schouten bracket as follows:

Definition 1.3 (Schouten bracket). For $A \in \Lambda^a \mathrm{T}(M)$ and $B \in \Lambda^b \mathrm{T}(M)$, define a binary operation $[A, B]_{\mathrm{S}} \in \Lambda^{a+b-1} \mathrm{T}(M)$ by

$$(-1)^{a+1} [A, B]_{\mathrm{S}} = \partial_0 (A \wedge B) - (\partial_0 A) \wedge B - (-1)^a A \wedge \partial_0 B \ , \tag{1.10}$$

where ∂_0 is the boundary operator in the context of Lie algebra homology of vector fields.

In this sense, the Schouten bracket measures the gap of the boundary operator ∂_0 from the derivation. Hereafter, we denote $[A, B]_{\mathrm{S}}$ simply by $[A, B]$.

For the pre Lie superalgebra $\mathfrak{g}$ (1.9) above, the chain spaces in lower degrees are described as follows. The first chain space is $\mathrm{C}_1 = \mathfrak{g} = \sum_{p=1}^{n} \Lambda^p \mathrm{T}(M)$. The second chain space is

$$\begin{aligned} \mathrm{C}_2 = \mathfrak{g} \triangle \mathfrak{g} &= \sum_{1 \leq p \leq q \leq n} \Lambda^p \mathrm{T}(M) \triangle \Lambda^q \mathrm{T}(M) \\ &= \Lambda^1 \mathrm{T}(M) \triangle \Lambda^1 \mathrm{T}(M) + \Lambda^1 \mathrm{T}(M) \triangle \Lambda^2 \mathrm{T}(M) + \cdots \\ &\quad + \Lambda^2 \mathrm{T}(M) \triangle \Lambda^2 \mathrm{T}(M) + \Lambda^2 \mathrm{T}(M) \triangle \Lambda^3 \mathrm{T}(M) + \cdots \end{aligned}$$

Remark 1.4. Let $\pi \in \Lambda^2 \mathrm{T}(M)$. Then $\pi \triangle \pi \in \Lambda^2 \mathrm{T}(M) \triangle \Lambda^2 \mathrm{T}(M) \subset \mathrm{C}_2$ and $\partial(\pi \triangle \pi) = [\pi, \pi] \in \mathrm{C}_1$. Thus, $\pi \in \Lambda^2 \mathrm{T}(M)$ is Poisson if and only if $\partial(\pi \triangle \pi) = 0$, and we express this phenomena symbolically by $\pi \in \sqrt{\ker(\partial)}$. It will be interesting to study $\sqrt{\ker(\partial)}$ and also interesting to study specific properties of Poisson structures in $\sqrt{\partial(\mathrm{C}_3)}$, which come from the boundary image of the third chain space C_3.

1.1 Weight

Let $\mathfrak{g} = \sum_i \mathfrak{g}_i$ be a pre Lie superalgebra.

Definition 1.5. We say a non-zero m-chain in $\mathfrak{g}_{i_1} \triangle \cdots \triangle \mathfrak{g}_{i_m}$ has the weight $i_1 + \cdots + i_m$. In particular, a non-zero 1-chain in $\mathfrak{g}_i$ has the weight i.

Definition 1.6. For a given w, we define the subspace $\mathrm{C}_{m,w} = \sum_{\sum_{s=1}^{m} i_s = w} \mathfrak{g}_{i_1} \triangle \cdots \triangle \mathfrak{g}_{i_m}$ of m-th chain space C_m, which is the direct sum of different types of spaces of m-chains, but the same weight w. To fix the direct sum notation of $\mathrm{C}_{m,w}$, we may choose such $\{i_s\}$ satisfying $\sum_{s=1}^{m} i_s = w$ with a restriction $i_1 \leq \ldots \leq i_m$ by the supersymmetry of superalgebra $\mathfrak{g}$.

Proposition 1.7 ([2]). *The weight w is preserved by ∂, that is, we have $\partial(C_{m,w}) \subset C_{m-1,w}$. Thus, for a fixed w, we have m-th homology group of weight w*

$$H_{m,w}(\mathfrak{g}, \mathbb{R}) = \ker(\partial : C_{m,w} \to C_{m-1,w}) / \partial(C_{m+1,w}) \text{ and } H_m(\mathfrak{g}, \mathbb{R}) = \sum_{w} H_{m,w}(\mathfrak{g}, \mathbb{R}) .$$

1.2 Double weight

Definition 1.8. Let $\mathfrak{g} = \sum_i \mathfrak{g}_i$ be a pre Lie superalgebra. Suppose that each subspace $\mathfrak{g}_i$ is decomposed into subspaces $\mathfrak{g}_{i,j}$ as $\mathfrak{g}_i = \sum_j \mathfrak{g}_{i,j}$. If $\{\mathfrak{g}_{i,j} \mid i, j \in \mathbb{Z}\}$ satisfy

$$[\mathfrak{g}_{i_1,j_1}, \mathfrak{g}_{i_2,j_2}] \subset \mathfrak{g}_{i_1+i_2, j_1+j_2} \quad \text{for each } i_1, i_2, j_1, j_2 , \tag{1.11}$$

then we call $\mathfrak{g} = \sum_i \mathfrak{g}_i = \sum_{i,j} \mathfrak{g}_{i,j}$ double graded.

DEFINITION 1.9 (Double weight). Let $\mathfrak{g} = \sum_i \mathfrak{g}_i = \sum_{i,j} \mathfrak{g}_{i,j}$ be a double graded pre Lie superalgebra. We say a non-zero m-chain in $\mathfrak{g}_{i_1,h_1} \triangle \cdots \triangle \mathfrak{g}_{i_m,h_m}$ has the primary weight $\sum_{s=1}^m i_s$, the secondary weight $\sum_{s=1}^m h_s$, and the double weight $(\sum_{s=1}^{m} i_s, \sum_{s=1}^{m} h_s)$.

REMARK 1.10. In particular, we may say 1-chains in $\mathfrak{g}_{i,j}$ have the double weight (i, j).

DEFINITION 1.11. For a given pair (w, h) of integers, we define doubly weighted m-th chain space by

$$\mathrm{C}_{m,w,h} = \sum_{\substack{\sum_{s=1}^m i_s = w \\ \sum_{s=1}^m h_s = h}} \mathfrak{g}_{i_1,h_1} \triangle \cdots \triangle \mathfrak{g}_{i_m,h_m} \ .$$

From the same reason as in Definition 1.6, we only need to deal with non-decreasing sequences $\{\ i_s\}$ so that $\sum_{s=1}^m i_s = w$ for the primary weight w.

PROPOSITION 1.12 ([2]). *We have $\partial(C_{m,w,h}) \subset C_{m-1,w,h}$, that is, the double weight (w, h) is preserved by ∂. Thus, we have (w, h)-weighted homology groups*

$$H_{m,w,h}(\mathfrak{g}, \mathbb{R}) = \ker(\partial : C_{m,w,h} \to C_{m-1,w,h})/\partial(C_{m+1,w,h}) \ ,$$

and we see

$$H_{m,w}(\mathfrak{g}, \mathbb{R}) = \sum_{h} H_{m,w,h}(\mathfrak{g}, \mathbb{R}) \ , \quad H_m(\mathfrak{g}, \mathbb{R}) = \sum_{w,h} H_{m,w,h}(\mathfrak{g}, \mathbb{R}) \ .$$

REMARK 1.13. Given a double graded pre Lie superalgebra $\mathfrak{g} = \sum_{i,j} \mathfrak{g}_{i,j}$, we have the 3 kinds of weight, primary, secondary, and double weight. If we use $\mathfrak{g}_i = \sum_j \mathfrak{g}_{i,j}$, then $\mathfrak{g} = \sum_i \mathfrak{g}_i$ has the weight which is the same value as the primary weight. So, we sometimes call the weight in Definition 1.5 the primary weight.

2 The second Betti number for double weight $w = h$ in $\mathbb{R}^n$

We already know the direct sum of sections of exterior bundles of the tangent bundle of a manifold M forms a pre Lie superalgebra with the Schouten bracket. In this section, we deal with the n-plain $\mathbb{R}^n$ as a manifold with the Cartesian coordinates $x_1, \ldots, x_n$. We denote each space of a-multi vector fields with b-homogeneous polynomials by $\mathfrak{X}^a_b(\mathbb{R}^n)$. Then

$$\mathfrak{g} = \sum_{i=0}^{n-1} \mathfrak{g}_i \quad \text{where} \quad \mathfrak{g}_i = \sum_{j=-1}^{\infty} \mathfrak{g}_{i,j} \quad \text{and} \quad \mathfrak{g}_{i,j} = \mathfrak{X}^{i+1}_{j+1}(\mathbb{R}^n) \tag{2.1}$$

is a pre Lie superalgebra defined in (1.9). Since the Schouten bracket in this case satisfies $[\mathfrak{g}_{i_1,j_1}, \mathfrak{g}_{i_2,j_2}] \subset \mathfrak{g}_{i_1+i_2,j_1+j_2}$, we get a double graded pre Lie superalgebra

$$\mathfrak{g} = \sum_{i=0}^{n-1} \sum_{j=-1}^{\infty} \mathfrak{g}_{i,j} = \sum_{a=1}^{n} \sum_{b=0}^{\infty} \mathfrak{X}^{a}_{b}(\mathbb{R}^n) \ . \tag{2.2}$$

If m-th chain space $\mathrm{C}_{m,w,w} \neq (0)$ then $0 \leq w \leq m(n-1)$, so $\mathrm{C}_{2,w,w}$ consists of $\mathfrak{X}^{a_1}_{b_1} \triangle \mathfrak{X}^{a_2}_{b_2}$ with $a_1 + a_2 = 2 + w$ and $b_1 + b_2 = 2 + w$, where w must satisfy $0 \leq w \leq 2(n-1)$.
Since the primary weight of $x^B \partial_A$ is $|A| - 1$, it holds

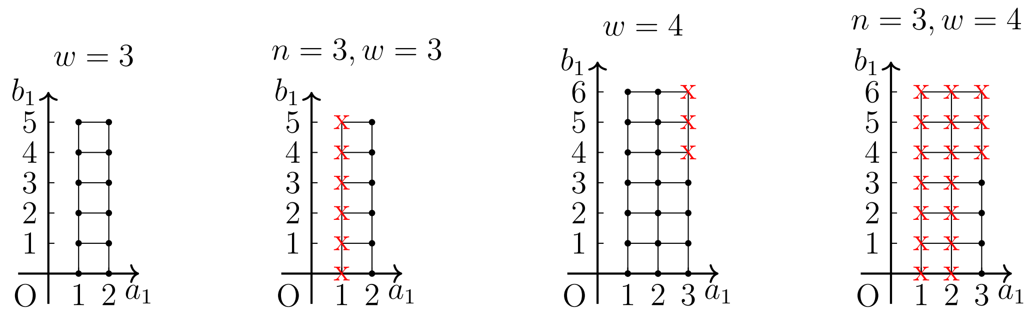
$$(x^B \partial_A) \triangle (x^{B'} \partial_{A'}) = -(-1)^{(|A|-1)(|A'|-1)} (x^{B'} \partial_{A'}) \triangle (x^B \partial_A) \ .$$

Since $\triangle$ has a super symmetric property, we have two ways to express the same space $\mathfrak{X}^{a_1}_{b_1} \triangle \mathfrak{X}^{a_2}_{b_2} = \mathfrak{X}^{a_2}_{b_2} \triangle \mathfrak{X}^{a_1}_{b_1}$, so we will express the space uniquely by setting $\mathrm{Term}^{a_1}_{b_1} := \mathfrak{X}^{a_1}_{b_1} \triangle \mathfrak{X}^{2+w-a_1}_{2+w-b_1}$ with $\mathtt{max}(1, 2+w-n) \leqq a_1 \leqq 1 + w/2$. We will denote $\mathtt{max}(1, 2+w-n)$ by the symbol hbd (n, w) or hbd (Horizontal left BounD).
We see that $\mathrm{C}_{2,w,w}$ has two expressions depending on w's parity.

$$\text{If } w = 2\Omega + 1, \qquad \mathrm{C}_{2,w,w} = \sum_{a_1=\mathrm{hbd}}^{\Omega+1} \sum_{b_1=0}^{2+w} \mathfrak{X}^{a_1}_{b_1} \triangle \mathfrak{X}^{2+w-a_1}_{2+w-b_1} \ .$$

$$\text{If } w = 2\Omega, \qquad \mathrm{C}_{2,w,w} = \sum_{a_1=\mathrm{hbd}}^{\Omega} \sum_{b_1=0}^{2+w} \mathfrak{X}^{a_1}_{b_1} \triangle \mathfrak{X}^{2+w-a_1}_{2+w-b_1} + \sum_{b_1=0}^{\Omega} \mathfrak{X}^{\Omega+1}_{b_1} \triangle \mathfrak{X}^{\Omega+1}_{2+w-b_1} + \mathfrak{X}^{\Omega+1}_{\Omega+1} \triangle \mathfrak{X}^{\Omega+1}_{\Omega+1} \ .$$

To understand how $\mathrm{C}_{2,w,w}$ is decomposed by $\mathrm{Term}^{a_1}_{b_1}$, we mark the point (a_1, b_1) on the 2-plane. The next are examples of $w = 3, 4$ in general, i.e., when hbd $= \mathtt{max}(1, 2+w-n) = 1$.



When $n = 3$, if $w = 3$ then we have to remove the line $a_1 = 1$ as in the picture above, and if $w = 4$ then we have to remove lines $a_1 = 1$ and $a_1 = 2$ as in the picture above.
When $n = 4$, if $w = 3$, then the picture we want is just the above. If $w = 4$ then we have to remove line $a_1 = 1$ of the picture above.

We introduce a total order on $\{ \, x^B \partial_A \mid A \in \{0,1\}^n, B \in \mathbb{N}^n \}$. First note that we have a one to one correspondence between the set $\{ \, x^B \partial_A \}$ and the set of polynomials $\{x^B y^A\}$,

and we have an order in the set of the polynomials by the graded reverse lexicographic order, where we put $x_1 < \cdots < x_n < y_1 < \cdots < y_n$. Precisely we compare $\{\, x^B \partial_A\}$ and $\{\, x^{B'} \partial_{A'}\}$ as follows: If $|A| < |A'|$ then define $x^B \partial_A < \, x^{B'} \partial_{A'}$. If $|A| = |A'|$ and $|B| < |B'|$ then define $x^B \partial_A < \, x^{B'} \partial_{A'}$. If $|A| = |A'|$ and $|B| = |B'|$ then use the comparison of $y^A x^B$ and $y^{A'} x^{B'}$ and accordingly $\{\, x^B \partial_A\}$and $\{\, x^B \partial_A\}$ are compared.
By a super symmetric property of $\triangle$, we may assume each single 2-chain is written as $x^{B_1} \partial_{A_1} \triangle \, x^{B_2} \partial_{A_2}$ with $x^{B_1} \partial_{A_1} \leqq \, x^{B_2} \partial_{A_2}$, and it is clear that any 2-chain is expressed as a linear combination of such single chains. We call such a chain *sorted*.

Definition 2.1. We define the type of each sorted 2-chain $x^{B_1} \partial_{A_1} \triangle \, x^{B_2} \partial_{A_2}$ by

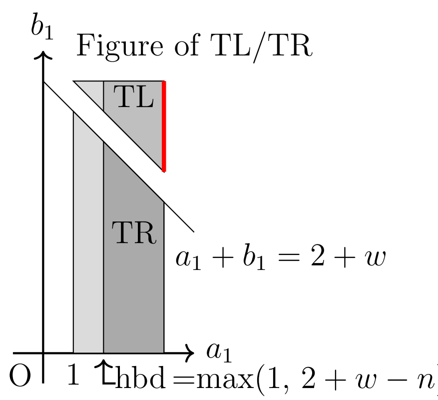
$$\begin{aligned} \text{type TR} \quad &\text{if} \quad |A_1| + |B_1| \leqq |A_2| + |B_2| \\ \text{type TL} \quad &\text{if} \quad |A_1| + |B_1| > |A_2| + |B_2| \; . \end{aligned}$$

If a sorted 2-chain $x^{B_1} \partial_{A_1} \triangle \, x^{B_2} \partial_{A_2} \in \mathrm{Term}^a_b$ is TR type, then each sorted 2-chain $x^{B'_1} \partial_{A'_1} \triangle \, x^{B'_2} \partial_{A'_2} \in \mathrm{Term}^a_b$ is also TR type. So we define a subspace Term^a_b is TR type if and only if one of sorted generators in Term^a_b is TR type. Similarly, we define a subspace Term^a_b is TL type if and only if one of sorted generators in Term^a_b is TL type.

To know the type of $\mathrm{Term}^{a_1}_{b_1}$ is easy:

$$\begin{aligned} &\text{TR type if and only if} \;\; a_1 + b_1 \leqq 2 + w \;, \text{ and} \\ &\text{TL type if and only if} \;\; a_1 + b_1 > 2 + w \; . \end{aligned}$$

If $a_1 + b_1 \leqq 2 + w$ and $a_1 < \mathrm{hbd}$ then $\mathrm{Term}^{a_1}_{b_1} = (0)$, i.e., some TR type is (0).



Remark 2.2. Naming TR comes from the Right side is bigger and TL means the Left side is bigger. Or, watching the rough picture "Figure of TL/TR" above, we imagine the line $a_1 + b_1 = 2 + w$ starting $(0, 2 + w)$ downward. TL type live in the left side of the direction, and TR type live in the right side.

Note that the double weights of $x^B \partial_A$ and $\partial_\ell \triangle x_\ell \, x^B \partial_A$ are equal, and those of $x^{B_1} \partial_{A_1} \triangle \, x^{B_2} \partial_{A_2}$ and $\partial_\ell \triangle x_\ell \, x^{B_1} \partial_{A_1} \triangle \, x^{B_2} \partial_{A_2}$ are equal, where $\partial_\ell = \frac{\partial}{\partial x_\ell}$ for each $\ell = 1, \ldots, n$. Noting this fact, we define the following two maps ϕ and Ψ.

Definition 2.3. Define a linear map $\phi : \mathrm{C}_{1,w,w} \to \mathrm{C}_{2,w,w}$ by

$$\phi(\, x^B \partial_A) = \sum_{\ell=1}^{n} \partial_\ell \triangle (x_\ell \, x^B \partial_A) \; . \tag{2.3}$$

For each sorted 2-chain $U = x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2}$, we define a 3-chain $\Phi(U)$ by

$$\sum_{\ell=1}^{n} \partial_\ell \triangle (x^{B_1}\partial_{A_1}) \triangle (x_\ell\, x^{B_2}\partial_{A_2}) \quad \text{if } U \text{ is of type TR} \tag{2.4}$$

and

$$\sum_{\ell=1}^{n} \partial_\ell \triangle (x_\ell\, x^{B_1}\partial_{A_1}) \triangle (x^{B_2}\partial_{A_2}) \quad \text{if } U \text{ is of type TL ,} \tag{2.5}$$

extending this linearly and we have $\Phi : \mathrm{C}_{2,w,w} \to \mathrm{C}_{3,w,w}$.

DEFINITION 2.4. Define a map $\Psi = \phi \circ \partial + \partial \circ \Phi : \mathrm{C}_{2,w,w} \to \mathrm{C}_{2,w,w}$ where it may be a part of "chain map" as follows:

$$\begin{array}{ccccccccc}
(0) & \xleftarrow{\partial} & \mathrm{C}_{1,w,w} & \xleftarrow{\partial} & \mathrm{C}_{2,w,w} & \xleftarrow{\partial} & \mathrm{C}_{3,w,w} & \xleftarrow{\partial} & \\
 & & \downarrow{\scriptstyle \partial\circ\phi} & \searrow{\scriptstyle \phi} & \downarrow{\scriptstyle \Psi} & \searrow{\scriptstyle \Phi} & & & \\
(0) & \xleftarrow{\partial} & \mathrm{C}_{1,w,w} & \xleftarrow{\partial} & \mathrm{C}_{2,w,w} & \xleftarrow{\partial} & \mathrm{C}_{3,w,w} & \xleftarrow{\partial} &
\end{array}$$

LEMMA 2.5. *Consider a sorted 2-chain* $x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2}$.

If it is of TR type, then

$$\begin{aligned}
\Psi(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2}) = & -\sum_{\ell=1}^{n} \partial_\ell \triangle [x^{B_1}\partial_{A_1}, x_\ell] x^{B_2}\partial_{A_2} + \sum_{\ell=1}^{n} [\partial_\ell, x^{B_1}\partial_{A_1}] \triangle (x_\ell x^{B_2}\partial_{A_2}) \\
& + (n + |B_2|)(x^{B_1}\partial_{A_1}) \triangle (x^{B_2}\partial_{A_2}) \ .
\end{aligned}$$

If it is of TL type, then

$$\begin{aligned}
\Psi(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2}) = & -(-1)^{|A_1|(|A_2|+1)} \sum_{\ell=1}^{n} \partial_\ell \triangle [x_\ell, x^{B_2}\partial_{A_2}] x^{B_1}\partial_{A_1} \\
& + (n + |B_1|)(x^{B_1}\partial_{A_1}) \triangle (x^{B_2}\partial_{A_2}) + \sum_{\ell=1}^{n} (x_\ell x^{B_1}\partial_{A_1}) \triangle [\partial_\ell, x^{B_2}\partial_{A_2}] \ .
\end{aligned}$$

Proof: When $x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2}$ is of TR type,

$$\begin{aligned}
& \partial(\Phi(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2})) = \partial(\sum_{\ell=1}^{n} \partial_\ell \triangle (x^{B_1}\partial_{A_1}) \triangle (x_\ell\, x^{B_2}\partial_{A_2})) \\
= & -\sum_{\ell=1}^{n} \partial_\ell \triangle [x^{B_1}\partial_{A_1}, x_\ell x^{B_2}\partial_{A_2}] + \sum_{\ell=1}^{n} [\partial_\ell, x^{B_1}\partial_{A_1}] \triangle (x_\ell x^{B_2}\partial_{A_2}) + \sum_{\ell=1}^{n} (x^{B_1}\partial_{A_1}) \triangle [\partial_\ell, x_\ell x^{B_2}\partial_{A_2}] \\
= & -\phi(\partial(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2})) - \sum_{\ell=1}^{n} \partial_\ell \triangle [x^{B_1}\partial_{A_1}, x_\ell] x^{B_2}\partial_{A_2} \\
& + \sum_{\ell=1}^{n} [\partial_\ell, x^{B_1}\partial_{A_1}] \triangle (x_\ell x^{B_2}\partial_{A_2}) + (x^{B_1}\partial_{A_1}) \triangle \sum_{\ell=1}^{n} (x^{B_2} + [x_\ell\partial_\ell, x^{B_2}])\partial_{A_2}
\end{aligned}$$

$$= -\phi(\partial(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2})) - \sum_{\ell=1}^{n} \partial_\ell \triangle [x^{B_1}\partial_{A_1}, x_\ell] x^{B_2}\partial_{A_2}$$
$$+ \sum_{\ell=1}^{n} [\partial_\ell, x^{B_1}\partial_{A_1}] \triangle (x_\ell x^{B_2}\partial_{A_2}) + (n + |B_2|)(x^{B_1}\partial_{A_1})\triangle(x^{B_2}\partial_{A_2}) \ ,$$

where, we used a property of the Euler vector field $\sum_{\ell=1}^{n} x_\ell \partial_\ell$ for polynomials.

When $x^{B_1}\partial_{A_1}\triangle x^{B_2}\partial_{A_2}$ is of TL type,

$$\partial(\Phi(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2})) = \partial(\sum_{\ell=1}^{n} \partial_\ell \triangle (x_\ell\, x^{B_1}\partial_{A_1}) \triangle (\, x^{B_2}\partial_{A_2}))$$
$$= -\sum_{\ell=1}^{n} \partial_\ell \triangle [x_\ell x^{B_1}\partial_{A_1}, x^{B_2}\partial_{A_2}] + \sum_{\ell=1}^{n} [\partial_\ell, x_\ell x^{B_1}\partial_{A_1}] \triangle (x^{B_2}\partial_{A_2}) + \sum_{\ell=1}^{n} (x_\ell x^{B_1}\partial_{A_1}) \triangle [\partial_\ell, x^{B_2}\partial_{A_2}]$$
$$= -\phi(\partial(x^{B_1}\partial_{A_1} \triangle x^{B_2}\partial_{A_2})) - (-1)^{|A_1|(|A_2|+1)} \sum_{\ell=1}^{n} \partial_\ell \triangle [x_\ell, x^{B_2}\partial_{A_2}] x^{B_1}\partial_{A_1}$$
$$+ (n + |B_1|)(x^{B_1}\partial_{A_1})\triangle(x^{B_2}\partial_{A_2}) + \sum_{\ell=1}^{n} (x_\ell x^{B_1}\partial_{A_1}) \triangle [\partial_\ell, x^{B_2}\partial_{A_2}] \ .$$

■

We deal with polynomail operators of Ψ. For instance $a\Psi^2 + b\Psi + c$ is a linear map defined by $(a\Psi^2 + b\Psi + c)(u) = a\Psi(\Psi(u)) + b\Psi(u) + cu$ for a, b, c are constant.
From Lemma 2.5 above, we have

PROPOSITION 2.6. *Take* Term^{a}_{b} *of* $C_{2,w,w}$. *Then*

$$(\Psi - (n + 2 + w - b))(\mathrm{Term}^{a}_{b}) \subset \mathrm{Term}^{a}_{b-1} + \mathrm{Term}^{1}_{0} \quad \textit{holds if } \mathrm{Term}^{a}_{b} \textit{ is TR type.} \tag{2.6}$$
$$(\Psi - (n + b))(\mathrm{Term}^{a}_{b}) \subset \mathrm{Term}^{a}_{b+1} + \mathrm{Term}^{1}_{0} \quad \textit{holds if } \mathrm{Term}^{a}_{b} \textit{ is TL type.} \tag{2.7}$$

The special subspace Term^{1}_{0} is of TR type, so using Lemma 2.5 directly, we have

PROPOSITION 2.7. Term^{1}_{0} *is invariant by* Ψ *and* $\Psi - (n + w + 1)$ *is identically zero on* Term^{1}_{0}.

REMARK 2.8. If hbd > 1, i.e., if $n < 1 + w$ then $\mathrm{Term}^{1}_{0} = (0)$. But Proposition 2.7 is still correct. So, we use Proposition 2.7 of general setting including $\mathrm{Term}^{1}_{0} = (0)$.
If $U \in \mathrm{Term}^{a}_{0}$ with $a > 1$, then it is TR type and $(\Psi - (n+2+w))U \in \mathrm{Term}^{a}_{-1} + \mathrm{Term}^{1}_{0} = \{0\} + \mathrm{Term}^{1}_{0} = \mathrm{Term}^{1}_{0}$ holds by (2.6).

2.1 Polynomial of Ψ vanishing on TR type

In this subsection, we find a polynomial of Ψ on $C_{2,w,w}$ which vanishes on the subspace of TR type elements. We recall $\text{hbd} = \max(1, 2+w-n)$, and put $\text{vbd} = 2+w-\text{hbd}$.

LEMMA 2.9. *We define*

$$P_0 = \prod_{j=0}^{\text{vbd}} (\Psi - (n+2+w-j)) \ . \tag{2.8}$$

Then $P = (\Psi - (n+w+1))P_0$ vanishes on the subspace of TR type in the 2-chain space $C_{2,w,w}$.

Proof: Consider a subspace Term^a_b of TR type, i.e., $a+b \leqq 2+w$. Note that $n+1 \leqq n+\text{hbd} \leqq n+2+w-b \leqq n+2+w$. (2.6) is

$$(\Psi - (n+2+w-b))\,\text{Term}^a_b \subset \text{Term}^a_{b-1} + \text{Term}^1_0 \ .$$

Since Term^a_{b-1} and Term^1_0 are type TR, using the (2.6) inductively, we get

$$\prod_{j=0}^{b} (\Psi - (n+2+w-(b-j)))\,\text{Term}^a_b \subset \text{Term}^1_0 \ . \tag{2.9}$$

The above polynomial of Ψ is equal to $\displaystyle\prod_{j=0}^{b}(\Psi - (n+2+w-j))$ by changing the index $b-j$ by j, and we denote the polynomial by $P^{[b]}$. Namely, we see

$$P^{[b]}\,\text{Term}^a_b \subset \text{Term}^1_0 \ . \tag{2.10}$$

Since $P^{[b]}$ is independent of a, we see that $P^{[b]}\,\text{Term}^{a'}_b \subset \text{Term}^1_0$ for any a' with $\text{Term}^{a'}_b$ of TR type.
If $b \leqq b'$ then $P^{[b']}\,\text{Term}^a_b \subset \text{Term}^1_0$, and vbd is the maxmum of those $\{b'\}$, and so we put $P^{[\text{vbd}]}$ by P_0. Then $P_0\,\text{Term}^a_b \subset \text{Term}^1_0$ for each Term^a_b of TR type. Using Lemma 2.7, we see that $(\Psi - (n+w+1))P_0\,\text{Term}^a_b = (0)$ for each Term^a_b of TR type, and we conclude $P = (\Psi - (n+w+1))P_0$ vanishes on the subspace of TR type in 2-chain space $C_{2,w,w}$. ■

REMARK 2.10. If $\text{hbd} > 1$ then $\text{Term}^1_0 = (0)$ and our discussion and conclusion in the proof above may become simpler. However if $\text{hbd} = 1$ then we have to handle non-trivial Term^1_0 and the proof above is in general setting including $\text{Term}^1_0 = (0)$.

2.2 Polynomial of Ψ vanishing on TL type

By an almost the same discussion as in the previous subsection, we find a polynomial of Ψ which vanishes on the subspace of TL type in the 2-chain space $C_{2,w,w}$. We recall Ω is used to describe w as $w = 2\Omega$ if w is even, or $w = 2\Omega + 1$ if w is odd.

LEMMA 2.11. *If we have non-trivial subspaces of TL type, then define*

$$Q_0 = \prod_{j=\Omega+3}^{2+w} (\Psi - (n+j)) \ . \tag{2.11}$$

Then $Q = (\Psi - (n+w+1))Q_0$ is a polynomial which vanishes on the subspace of TL type in the 2-chain space $C_{2,w,w}$.

Proof: Consider a subspace Term_b^a of TL type, i.e., $a + b > 2 + w$. Take any $U \in \mathrm{Term}_b^a$. Then using (2.7), we see

$$(\Psi - (n+b)) \, \mathrm{Term}_b^a \subset \mathrm{Term}_{b+1}^a + \mathrm{Term}_0^1 \ .$$

Even though Term_0^1 is type TR, it is invariant by Ψ and satisfies $\Psi \, \mathrm{Term}_0^1 \subset \mathrm{Term}_0^1$, and since Term_{b+1}^a is type TL, we get

$$\prod_{j=0}^{2+w-b} (\Psi - (n+b+j)) \, \mathrm{Term}_b^a \subset \mathrm{Term}_0^1$$

by induction. The above polynomial of Ψ is equal to $\displaystyle\prod_{j=b}^{2+w}(\Psi - (n+j))$ by changing the index $b + j$ by j, and we denote the polynomial by $Q^{[b]}$. Namely,

$$Q^{[b]} \, \mathrm{Term}_b^a \subset \mathrm{Term}_0^1 \ . \tag{2.12}$$

Since $Q^{[b]}$ is independent of a, we see that $Q^{[b]} \, \mathrm{Term}_b^{a'} \subset \mathrm{Term}_0^1$ for any a' with $\mathrm{Term}_b^{a'}$ of TL type. Let $b_0 = \mathtt{min}\{b' \mid Q^{[b']} \, \mathrm{Term}_b^a \subset \mathrm{Term}_0^1\}$. From the picture "Figure of TL/TR" suggests the longest vertical segment of TL type lies on the line $a_1 = \Omega$ if $w = 2\Omega + 1$ or $a_1 = \Omega + 1$ if $w = 2\Omega$, and $b_0 = \Omega + 3$. Define $Q_0 = Q^{[b_0]} = Q^{[\Omega+3]}$. Then $Q_0 \, \mathrm{Term}_b^a \subset \mathrm{Term}_0^1$ for each Term_b^a of TL type. Using Lemma 2.7, we see that

$$(\Psi - (n+w+1))Q_0 \, \mathrm{Term}_b^a = (0) \ \text{ for } \ \mathrm{Term}_b^a \ \text{ of TL type} \ .$$

Thus $Q = (\Psi - (n+w+1))Q_0$ vanishes on the subspace of TL type in the 2-chain space $C_{2,w,w}$. ■

2.3 Main Result

Our main result is the following:

THEOREM 2.12. *The second Betti number is zero for* $\{C_{\bullet,w,w}\}$.

Proof of Theorem: If we have non-trivial subspaces of TL type, from Lemma 2.9 and Lemma 2.11, the polynomial of Ψ

$$(\Psi - (n+w+1))P_0\, Q_0 = (\Psi - (n+w+1)) \prod_{j=0}^{\text{vbd}} (\Psi - (n+2+w-j)) \prod_{j=\Omega+3}^{2+w} (\Psi - (n+j))$$

vanishes on $\mathrm{C}_{2,w,w}$.
We put this polynomial by $(\Psi + c_1)\cdots(\Psi + c_m)$ for simplicity. The key point is that each c_i is non-zero number. $(\Psi + c_1)\cdots(\Psi + c_m)$ is identically zero on $\mathrm{C}_{2,w,w}$ and we expand the polynomial as $c_1 \cdots c_m + \Psi g(\Psi)$ for some polynomial g of Ψ. We see that

$$c_1 \cdots c_m U + \Psi g(\Psi) U = 0 \quad \text{for} \quad U \in \mathrm{C}_{2,w,w}\ . \tag{2.13}$$

Now assume U is a cycle. Then $\Psi(U) = (\partial \circ \Phi + \phi \circ \partial)U = (\partial \circ \Phi)U$ and from (2.13) we get

$$0 = c_1 \cdots c_m U + \partial \circ \Phi \circ g(\partial \circ \Phi) U\ . \tag{2.14}$$

This implies U is exact.
If we have no subspace of TL type, then Lemma 2.9 implies the polynomial of Ψ

$$(\Psi - (n+w+1))P_0 = (\Psi - (n+w+1)) \prod_{j=0}^{\text{vbd}} (\Psi - (n+2+w-j))$$

vanishes on $\mathrm{C}_{2,w,w}$. The same discussion as above shows that any 2-cycle in $\mathrm{C}_{2,w,w}$ is exact. ■

Concerning Remark 1.4, the following corollary holds, which is a direct consequence of Theorem 2.12.

COROLLARY 2.13. *Each 2-vector field* $\pi \in \mathfrak{X}^2_0$ *of constant coefficients is Poisson.* $\pi \in \mathfrak{X}^2_2$ *is a Poisson 2-vector field if and only if* $\pi \triangle \pi = \partial \tau$ *for some* $\tau \in C_{3,\,2,\,2}$.

REMARK 2.14. This corollary fills the gap of the result in [2] that for each positive integer $p \neq 2$, $\pi \in \mathfrak{X}^2_p$ is a Poisson 2-vector field if and only if $\pi \triangle \pi = \partial \tau$ for some $\tau \in \mathrm{C}_{3,\,2,\,2(p-1)}$.

References

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