トポロジーを暗号に応用したい
How to Apply Topology to Cryptology, Hopefully

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Commuting Operations

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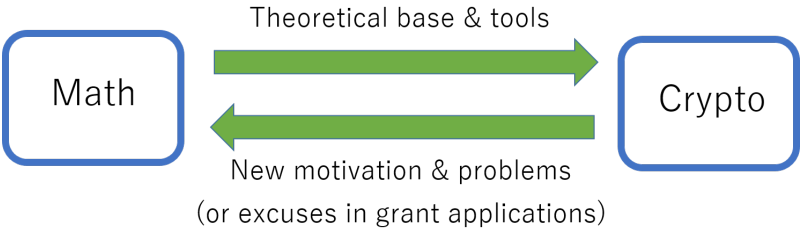
Relation between Mathematics and Cryptography



Q. How about topology?

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- $\mathrm{Dec}_{\mathrm{sk}}(\mathrm{Enc}_{\mathrm{pk}}(m)) = m$
- pk **should not yield information on** sk

The RSA Cryptosystem [1977 ?]

- $N = pq$ (distinct primes)
- e, d with $ed \equiv 1 \pmod{(p-1)(q-1)}$

Given message $m \in (\mathbb{Z}/N\mathbb{Z})^\times$,

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Drawback: Enc is deterministic ("textbook RSA")

- Improved variant is practically used

Computational Assumptions

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- E.g. "Factoring N is hard" for the RSA
- Theoretically, just "assumption" (cf. P vs NP)
    - Practically, evaluated by experiments
    - Consensus: "(General) Number Field Sieve" would factorize $N \approx 2^{1024}$ in near future

Prior to RSA — Diffie–Hellman Key Exchange [1976]

Protocol between parties P_1 and P_2

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2.

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- Can be converted to PKE [ElGamal 1985]

Is DH Key Exchange Secure?

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Public: $G = \langle g \rangle$ and $h_i \in G$
Secret: a_i with $h_i = g^{a_i}$

$\Rightarrow$ The **discrete logarithm problem** (DL) in G must be computationally hard:

(DL) $\boxed{\text{Given } g, h, \text{ find } x \text{ with } h = g^x \text{ in } G}$

- Remark: (In)sufficiency is still open

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$\Rightarrow$ Difficulty of DL **does depend** on
a "realization" of the same abstract group G!

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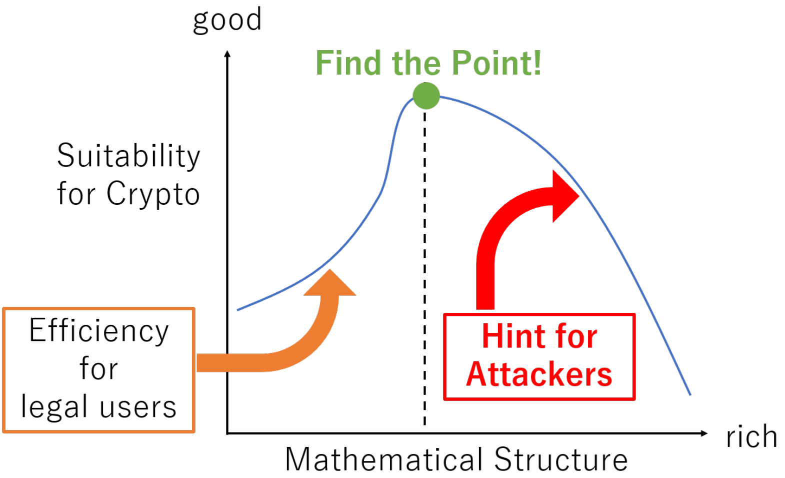
A lesson: Additional structure for group G makes the DL easier ($\Rightarrow$ break of DH Key Exchange)

- Cf. [Maurer 2005] DL is hard in "generic group"
  - "Oracle access to multiplication table only"

A New Viewpoint from Cryptography



(Mathematician: more structures, more happiness)

State-of-the-Art Candidate from Elliptic Curves

(Subgroups of) groups of rational points on **elliptic curves** (over finite fields)

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Current status: $|G| \gtrapprox 2^{160}$

- Cf. $N \gtrapprox 2^{1024}$ for the RSA

Shor's Quantum Algorithms

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- integer factoring **in polynomial time**!
- discrete logarithm **in polynomial time**!

(Cf. [Grover 1996]: Search with quadratic speedup)

Pinpoint Effect by Shor to Cryptography

Shor's main applications: integer factoring and DL

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$\longrightarrow$ Importance of "quantum-resistant" PKE

- (Believed to be) unbroken by quantum computer

Candidates of quantum-resistant PKE

Based on (conjectural) hardness of solving:

- Knapsack problem
- System of multivariate quadratic equations
- Decoding random linear codes
- Shortest vectors in integer lattices
- Finding sections on algebraic surfaces
- Finding isogeny between elliptic curves
- ...

Candidates of quantum-resistant PKE

Based on (conjectural) hardness of solving:

- Knapsack problem (not good)
- System of multivariate quadratic equations (fair)
- Decoding random linear codes (sometimes good)
- Shortest vectors in integer lattices (hopeful)
- Finding sections on algebraic surfaces (?)
- Finding isogeny between elliptic curves (?)
- ...

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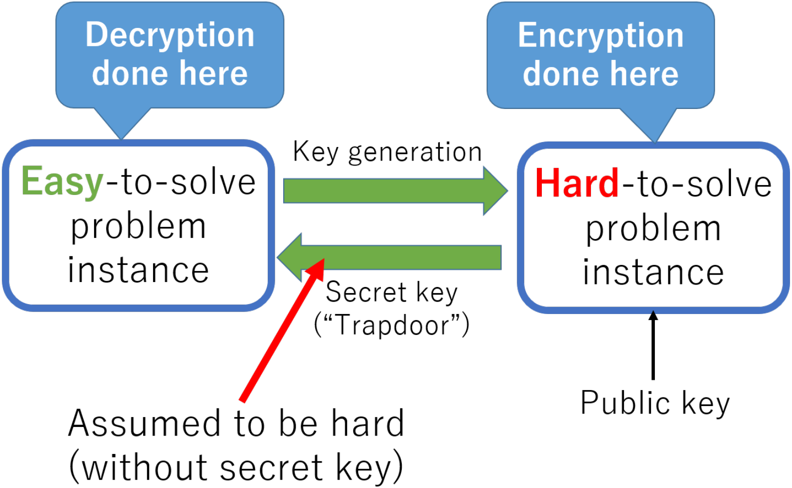
“Security of PKE” means

- “**Almost all**” ciphertexts are hard to break

A Major Strategy for PKE



Multi-Party Computation (MPC): Two-Party Case

Given function $f(x, y)$ (e.g., $f(x, y) = \delta_{x,y}$),

- Party P_1 has secret input a_1
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- while **hiding information on each input!**
  - (except those trivially implied from $f(a_1, a_2)$)

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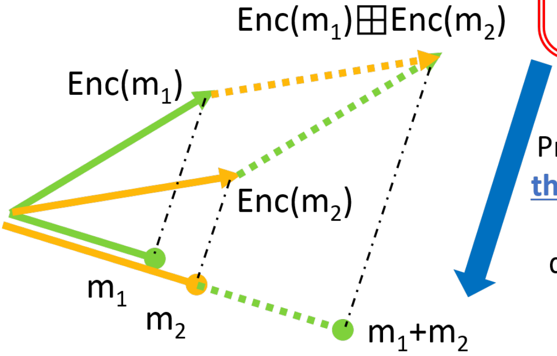
- "Messages can be added in encrypted form"

A "Rough Idea" for HE



Direction should be hard to find without key (art of cryptographer)

Projection in **this direction** yields decryption

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- Homomorphic operation: multiplication in G^2

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    - by random iteration of $\boxplus$ to $[[a_1 - a_2]]$
5. P_1 decrypts $[[r(a_1 - a_2)]] \rightsquigarrow 0$ iff $a_1 = a_2$

Fully Homomorphic Encryption [2009]

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- $\Leftrightarrow$ Ring-HE, when $\mathcal{M} = \mathbb{F}_p$ (p prime)

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$\mathbb{Z}/\ell\mathbb{Z}$ identified with $\{0, \ldots, \ell - 1\}$ by "mod"

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"Bootstrapping": refreshing the ciphertext

- possible, but very inefficient

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$\rightsquigarrow$ hard-to-compute group hom. $\varphi \colon \widetilde{G} \twoheadrightarrow G$

- must be easy-to-compute with secret key
- Public: G and generators of $\ker \varphi$ (for Enc)

How to "Realize" $\mathbb{F}_2$ in $\mathrm{PSL}_2(\mathbb{F}_q)$

[N. 2014 (preprint)]

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For $c, c' \in X_0 \cup X_1$, define (with random $g \in G$)

$$[c, c']^{\dagger} := ([g^{-1} c_1 g, c'_1], [g^{-1} c_2 g, c'_2])$$

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With probability $\approx 1 - q^{-1}$ we have: ($\rightsquigarrow$ "AND")

- If $c, c' \in X_1$, then $[c, c']^\dagger \in X_1$
- Otherwise, $[c, c']^\dagger \in X_0$

How to Realize NAND Gate in Simple Groups

$\mathrm{NAND}(b_1, b_2) = 0$ iff $b_1 = b_2 = 1$
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(Proof idea: $\langle \text{commutators} \rangle_{\mathrm{normal}} = G$)

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Towards Homomorphically Hiding the Group

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Current problems:

- Knuth-Bendix algorithm may not terminate
- Is it really secure?

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- Embedding into other topology-related objects? (E.g., quandles from knot theory)